

EULER-LAGRANGE FORMULAS FOR PSEUDO-KÄHLER MANIFOLDS

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ABSTRACT. Let c be a characteristic form of degree k which is defined on a Kähler manifold of real dimension $m > 2k$. Taking the inner product with the Kähler form Ω^k gives a scalar invariant which can be considered as a generalized Lovelock functional. The associated Euler-Lagrange equations are a generalized Einstein-Gauss-Bonnet gravity theory; this theory restricts to the canonical formalism if $c = c_2$ is the second Chern form. We extend previous work studying these equations from the Kähler to the pseudo-Kähler setting.

1. INTRODUCTION

1.1. Historical perspective. The Euler form was introduced by Chern [3] in his generalization of the Gauss-Bonnet theorem to higher dimensions. Let R_{ijkl} be the components of the Riemann curvature tensor. Let $m = 2\bar{m}$ be even. The Pfaffian or Euler integrand is defined to be

$$E_m(g) = \frac{1}{(8\pi)^{\bar{m}}\bar{m}!} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{m-1} i_m j_m j_{m-1}} g(e^{i_1} \wedge \cdots \wedge e^{i_m}, e^{j_1} \wedge \cdots \wedge e^{j_m})$$

where we adopt the *Einstein convention* and sum over repeated indices. Let $\text{dvol}_g := \sqrt{\det(g_{ij})} dx^1 \cdots dx^m$ be the Riemannian measure. Chern showed that if M is a compact Riemannian manifold of dimension m without boundary, then the Euler-Poincaré characteristic $\chi(M)$ is given in terms of curvature:

$$\chi(M^m) = \int_M E_m(g) \text{dvol}_g .$$

This generalizes the classical Gauss-Bonnet formula from 2 dimensions to the higher dimensional setting. Subsequently, Chern [4] introduced the so called *Chern classes*; these will be discussed in more detail in the next section. Let $c_{\bar{m}}$ be the \bar{m}^{th} Chern class. If (M, g, J) is a Kähler manifold, then $E_m(g) = c_{\bar{m}}(g)$ so this particular characteristic class reproduces the Euler form. The theory of characteristic classes is, of course, much more general and plays an important role in the Hirzebruch-Riemann-Roch Theorem [12] amongst many applications.

Alternative theories of gravity arise naturally in physics since the standard particle model and general relativity seem to fail at extreme regimes of ultra violet scales. Lovelock [19] introduced Chern-Gauss-Bonnet gravity by studying the Euler-Lagrange equations associated with $E_4 = \frac{1}{32\pi^2} \{\tau^2 - 4|\rho|^2 + |R|^2\}$ in order to study the Einstein field equations in vacuo; it is a non-linear theory of gravity. The action is crucial in dimensions $m > 4$. These equations (with appropriate perturbing terms) have been used by D. Chirkov, S. Pavluchenko and A. Toporensky [5, 6]

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to investigate the constant volume exponential solutions in the Einstein-Gauss-Bonnet gravity in $4 + 1$ and $5 + 1$ dimensional space times; the metric is, of course, Lorentzian. Work by Najian [20] treats aspects of holographic dual of boundary conformal field theories for higher derivative Gauss-Bonnet (GB) gravity. Nozari et al. [21] treat a DGP-inspired braneworld model such that the induced gravity on the brane uses a bulk action which contains the Gauss-Bonnet term to incorporate higher order curvature effects. Zeng and Liu [22] study the thermalization of a dual conformal field theory to Gauss-Bonnet gravity by modeling a thin-shell of dust that interpolates between a pure AdS and a Gauss-Bonnet AdS black brane.

The Euler-Lagrange equations have also been used [13] by J. B. Jimenez and T. S. Koivisto in the context of Weyl geometry to investigate an extended Gauss-Bonnet gravity theory in arbitrary dimensions and in a space provided with a Weyl connection. See also work of D. Butter et al. [2] on the Gauss-Bonnet density in superspace and work of W. Yao and J. Jing [23] dealing with a Born-Infeld electromagnetic field coupling with a charged scalar field in the five-dimensional Einstein-Gauss-Bonnet spacetime. The field is a vast one and we can not do justice to it with this brief summary.

The higher dimensional Euler integrands are important; for example Dadhich and Pons [7] examine Lovelock functional \mathcal{L}_{2k} for arbitrary k in Lorentzian signature. Although a-priori the Euler-Lagrange equations for the Lovelock functional defined by E_m can involve the 4th derivatives of the metric in dimensions $n > m$, Berger [1] conjectured it only involved curvature; this was subsequently verified by Kuz'mina [14] and Labbi [15, 16, 17]. Following de Lima and de Santos [18], one says that a compact Riemannian n -manifold is $2k$ -Einstein for $2 \leq 2k < n$ if it is a critical metric for the Einstein-Hilbert-Lovelock functional $L_{2k}(g) = \int_M E_{2k} \text{dvol}$ when restricted to metrics on M with unit volume. This involves, of course, examining the associated Euler-Lagrange equations for this functional. Indefinite signatures are also important; Gilkey, Park and Sekigawa examined the Euler-Lagrange equations in the higher signature setting [10].

In previous work [11], we extended these results for the Lovelock functional to the setting of characteristic classes in the Kähler context; in higher dimensions it seemed possible that appropriate gravity theories could be based on arbitrary characteristic classes and not just on the Lovelock functional. We gave explicit formulas for the appropriate Euler-Lagrange equations and showed that the map from the characteristic forms to the symmetric 2-tensors given by the Euler-Lagrange equations coincides with the map given algebraically by the transgression in the Kähler setting – see Theorem 1 below for details. It is the purpose of this present paper to extend these results to the indefinite setting with a minimum of technical fuss and in particular not to repeat the analysis of [11] but rather to use analytic continuation as indefinite signatures playing a crucial role in many applications; there are string theories where the hidden dimensions also have indefinite signatures.

1.2. A review of Chern-Weil Theory and the characteristic classes. Let V be a real vector bundle of dimension 2ℓ which is equipped with an almost complex structure J . We use J to give V the structure of a complex vector bundle V_c by defining $\sqrt{-1}v := Jv$. Let ∇ be a connection on V which commutes with J . Since J then commutes with the curvature R of ∇ , we may regard R as a complex 2-form valued endomorphism R_c of V_c . Let $\mathfrak{C}_{k,\ell}$ be the collection of polynomial maps $\Theta(\cdot)$ from the space of $\ell \times \ell$ complex matrices $M_\ell(\mathbb{C})$ to \mathbb{C} which are homogeneous of degree k and which satisfy $\Theta(gAg^{-1}) = \Theta(A)$ for all $A \in M_\ell(\mathbb{C})$ and all g in the general linear group $GL_\ell(\mathbb{C})$. If $\Theta \in \mathfrak{C}_{k,\ell}$, we may define $\Theta(R_c) \in \Lambda^{2k}(M) \otimes_{\mathbb{R}} \mathbb{C}$ invariantly (i.e. independent of the particular local frame chosen for V_c). One has that $\Theta(R_c)$ is a closed $2k$ -form and the de Rham cohomology class of $\Theta(R_c)$ is

independent of the particular connection chosen; these are the celebrated *characteristic classes* of Chern [4]. Other structure groups, of course, give rise appropriate characteristic classes; the Pontrjagin classes, for example, relate to the orthogonal group while the Euler form E_m can properly be regarded as an characteristic class of the special orthogonal group.

1.3. Holomorphic Geometry. Let M be a smooth manifold of (real) dimension $m := 2\bar{m}$. We say that J is an *integrable complex structure* on the tangent bundle TM if there is a coordinate atlas with local coordinates (x^1, \dots, x^m) so that

$$J\partial_{x_i} = \partial_{x_{i+\bar{m}}} \quad \text{and} \quad J\partial_{x_{i+\bar{m}}} = -\partial_{x_i} \quad \text{for} \quad 1 \leq i \leq \bar{m}. \quad (1.a)$$

Since $J^2 = -\text{id}$, we may use J to give TM a complex structure and regard TM as a complex vector bundle. We say that a pseudo-Riemannian metric h on TM is *pseudo-Hermitian* if $J^*h = h$. Let ∇^h be the Levi-Civita connection of h . Since ∇^h need not commute with J , we form the connection

$$\tilde{\nabla}^h := \frac{1}{2}\{\nabla^h + J^*\nabla^h\} = \frac{1}{2}\{\nabla^h - J\nabla^h J\}.$$

Let R_c^h be the associated complex curvature tensor. If $\Theta \in \mathfrak{C}_{k,\ell}$, then we may form

$$\Theta(h) := \Theta(R_c^h) \in \Lambda^{2k}(M) \otimes_{\mathbb{R}} \mathbb{C}.$$

Of particular interest is the special case where the Levi-Civita connection actually does commute with J , i.e. $\nabla^h(J) = 0$ and the triple (M, h, J) is said to be a *pseudo-Kähler* manifold (if h is positive definite, then (M, h, J) is said to be a *Kähler* manifold). The pseudo-Kähler geometries are very special as we shall see presently. One feature is that there exist normal-holomorphic coordinates, i.e. holomorphic coordinate systems where the first derivatives of the metric vanish. We refer to Section 2 for details. If (M, h, J) is a pseudo-Kähler manifold, then $\tilde{\nabla}^h = \nabla^h$. Let $c_\ell(A) := \det(\frac{\sqrt{-1}}{2\pi}A)$; then $c_\ell \in \mathfrak{C}_{\ell,\ell}$ and $c_\ell(h) = E_{2\ell} \text{dvol}$ is the integrand of the Chern-Gauss-Bonnet theorem.

1.4. Euler-Lagrange equations associated to the characteristic classes. Let $\Omega_h(x, y) := h(x, Jy)$. We assume (M, h, J) is pseudo-Kähler; this implies $d\Omega_h = 0$. Any complex manifold inherits a natural orientation so we may identify measures with m -forms. In the positive definite setting, we have under this identification that $\text{dvol} = \frac{1}{\bar{m}!}\Omega_h^{\bar{m}}$ and we replace dvol by $\frac{1}{\bar{m}!}\Omega_h^{\bar{m}}$ to avoid complications with signs henceforth in the higher signature context.

Let $\Theta \in \mathfrak{C}_{k,\ell}$. As noted above, $\Theta(h)$ is a differential form of degree $2k$. To obtain a scalar invariant, we contract with the Kähler form $\Omega_h(x, y) := h(x, Jy)$ and in analogy with Gauss-Bonnet gravity (as discussed above), we consider the scalar invariant $h(\Theta(h), \Omega_h^k)$ and associated Lovelock functional

$$\Theta[M, h, J] := \frac{1}{\bar{m}!} \int_M h(\Theta(h), \Omega_h^k) \cdot \Omega_h^{\bar{m}}.$$

The associated Euler-Lagrange equations are defined by setting:

$$\text{EL}_\Theta(h, \kappa) := \frac{1}{\bar{m}!} \left\{ \partial_\epsilon \int_M h(\Theta(h + \epsilon\kappa), \Omega_{h+\epsilon\kappa}^k) \cdot \Omega_{h+\epsilon\kappa}^{\bar{m}} \right\} \Big|_{\epsilon=0}$$

where κ is a J -invariant symmetric 2-tensor with compact support. If $\bar{m} = k$ and if M , then $\Theta[M, h, J]$ is independent of h and hence the associated Euler-Lagrange equations vanish. We therefore suppose that $k < \bar{m}$. We can integrate by parts to express

$$\text{EL}_\Theta(h, \kappa) = \frac{1}{\bar{m}!} \int_M \langle \mathcal{E}_\Theta(h), \kappa \rangle \cdot \Omega_h^{\bar{m}}$$

where $\mathcal{E}_\Theta(h) \in S^2(TM)$ is a J -invariant symmetric 2-tensor field and where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $S^2(TM)$ and $S^2(T^*M)$. Let

$$\mathcal{F}_\Theta(h) := -\frac{1}{k+1}\sqrt{-1}h((\Theta(R_c^h) \wedge e^\alpha \wedge e^{\bar{\beta}}) \otimes e_\alpha \otimes e_{\bar{\beta}}).$$

Of course, for a general h , $\mathcal{E}_\Theta(h)$ is very complicated and can not be expressed directly in terms of curvature. However, after a lengthy and difficult calculation in invariance theory, it was shown [11] that

Theorem 1. *Let $\Theta \in \mathfrak{C}_{k,\bar{m}}$. If (M, h, J) is a Kähler manifold, then*

$$\mathcal{E}_\Theta(h) = \mathcal{F}_\Theta(h).$$

The main result of this brief note extends Theorem 1 to the pseudo-Kähler setting.

Theorem 2. *Let $\Theta \in \mathfrak{C}_{k,\bar{m}}$. If (M, h, J) is a pseudo-Kähler manifold, then*

$$\mathcal{E}_\Theta(h) = \mathcal{F}_\Theta(h).$$

One could redo the analysis of [11] taking into account the fact that the structure group $U(p, q)$ involves not only rotations but also hyperbolic boosts. But instead, we will use analytic continuation to pass from the positive definite to the indefinite setting. Such methods of analytic continuation have been previously; see, for example, the discussion in García-Río et al. [9] that spacelike and timelike Osserman are equivalent concepts. In Section 2, we will review some results concerning pseudo-Kähler geometry which we shall need and we shall give the proof of Theorem 2 using analytic continuation.

2. PSEUDO-KÄHLER GEOMETRY

Let (M, h, J) be a pseudo-Kähler manifold of signature $(2\bar{p}, 2\bar{q})$. Fix a point $P \in M$ and choose local coordinates $x = (x^1, \dots, x^m)$ centered at P so that J is given by Equation (1.a). We complexify and set, for $1 \leq \alpha \leq \bar{m}$,

$$\begin{aligned} \partial_{z^\alpha} &:= \frac{1}{2}\{\partial_{x^\alpha} - \sqrt{-1}\partial_{x^{\bar{m}+\alpha}}\}, & dz^\alpha &:= \{dx^\alpha + \sqrt{-1}dx^{\bar{m}+\alpha}\}, \\ \partial_{\bar{z}^\alpha} &:= \frac{1}{2}\{\partial_{x^\alpha} + \sqrt{-1}\partial_{x^{\bar{m}+\alpha}}\}, & d\bar{z}^\alpha &:= \{dx^\alpha - \sqrt{-1}dx^{\bar{m}+\alpha}\}. \end{aligned}$$

We extend the metric h to be complex bilinear and set $h_{\alpha,\bar{\beta}} := h(\partial_{z^\alpha}, \partial_{\bar{z}^\beta})$. The condition that $J^*h = h$ is then equivalent to the identities: The condition that $J^*h = h$ then is reflected by the identities

$$h(\partial_{z^\alpha}, \partial_{z^\beta}) = 0, \quad h(\partial_{\bar{z}^\alpha}, \partial_{\bar{z}^\beta}) = 0, \quad \bar{h}(\partial_{z^\alpha}, \partial_{\bar{z}^\beta}) = h(\partial_{z^\beta}, \partial_{\bar{z}^\alpha}).$$

We set $h_{\alpha,\bar{\beta}} := h(\partial_{z^\alpha}, \partial_{\bar{z}^\beta})$. We then have $\bar{h}_{\alpha,\bar{\beta}} = h_{\beta,\bar{\alpha}}$. If we set $h_{\alpha,\bar{\beta}/\gamma} := \partial_{z^\gamma} h_{\alpha,\bar{\beta}}$ and $h_{\alpha,\bar{\beta}/\bar{\gamma}} := \partial_{\bar{z}^\gamma} h_{\alpha,\bar{\beta}}$, the Kähler condition becomes:

$$h_{\alpha,\bar{\beta}/\gamma} = h_{\gamma,\bar{\beta}/\alpha} \quad \text{and} \quad h_{\alpha,\bar{\beta}/\bar{\gamma}} = h_{\alpha,\bar{\gamma}/\bar{\beta}}. \quad (2.a)$$

Let $A := (\alpha_1, \dots, \alpha_\nu)$ and $B := (\beta_1, \dots, \beta_\mu)$ are a collection of indices between 1 and \bar{m} , we define

$$h(A; B) := \partial_{z^{\alpha_2}} \dots \partial_{z^{\alpha_\nu}} \partial_{\bar{z}^{\beta_2}} \dots \partial_{\bar{z}^{\beta_\mu}} h_{\alpha_1, \bar{\beta}_1}.$$

It is immediate that $\bar{h}(A; B) = h(B; A)$. We differentiate Equation (2.a) to see that we can permute the elements of A and that we can permute the elements of B without changing $h(A; B)$. The following Lemma was proved in [11] in the positive definite setting. The proof, however, involved quadratic and higher order holomorphic changes and was independent of the signature of the metric. It extends without change to the setting at hand.

Lemma 3. *Let P be a point of a Kähler manifold $M = (M, J, h)$. Fix $n \geq 2$. There exist local holomorphic coordinates $\vec{x} = (x^1, \dots, x^{2m})$ centered at P so that*

- (1) J is given by Equation (1.a).
- (2) $h(A; B)(P) = 0$ for $|B| = 1$ and $2 \leq |A| \leq n$.

Fix $n \geq 2$. Let constants $c(A; B) \in \mathbb{C}$ be given for $2 \leq |A| \leq n$ and $2 \leq |B| \leq n$ so that $c(A; B) = \bar{c}(B; A)$. There a Kähler metric \tilde{h} on (M, J) so that $\tilde{h}(A; B) = 0$ for $|B| = 1$ and so that $\tilde{h}(A; B)(P) = c(A; B)$ for $2 \leq |A| \leq n$ and $2 \leq |B| \leq n$.

The variables $\{h(A; B)\}$ are a good choice of variables since, unlike the covariant derivatives of the curvature tensor, there are no additional identities and we are dealing with a pure polynomial algebra.

Proof of Theorem 2. Fix $\Theta \in \mathfrak{C}_{k, \bar{m}}$. We work purely formally. We use Lemma 3 to regard \mathcal{E}_Θ and \mathcal{F}_Θ as polynomials in $h(P)$, the variables $h(A; B)$, and $\det(h)^{-1}$ where h is a J -invariant symmetric bilinear form on \mathbb{R}^m ; we must introduce the variable $\det(h)^{-1}$ to define the metric on the cotangent bundle and to raise and lower indices. These polynomials are well defined if $\det(h) \neq 0$ and we have $\mathcal{E}_\Theta - \mathcal{F}_\Theta = 0$ if h is positive definite.

But, of course, we can allow h to be complex valued. We have $\mathcal{E}_\Theta - \mathcal{F}_\Theta = 0$ if h is real valued and positive definite. Imposing the condition $\det(h) \neq 0$ does not disconnect the parameter space and thus the identity theorem yields $\mathcal{E}_\Theta - \mathcal{F}_\Theta = 0$ in complete generality and, in particular, if h has indefinite signature. \square

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